JOURNAL OF COMPUTATIONAL PHYSICS 60, 478–488 (1985)

Effect of Couple Stresses on Transient MHD Poiseuille Flow

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Received February 1, 1984; revised September 18, 1984

The unsteady laminar flow of an electrically conducting viscous fluid between parallel insulating plates subject to a transverse magnetic field is considered. The plates are fixed and flow is due to a constant pressure gradient. The induced field is taken into account. The fluid is incompressible and of couple stress type. The defining equations are coupled and numerical solutions for different values of couple stress parameter are obtained. The velocity and induced magnetic field profiles are sketched as functions of time, Hartmann number, and magnetic Prandtl number. The velocity decreases with increase in couple stress parameter.

1. INTRODUCTION

In conventional MHD, Poiseuille flow has been the subject of many investigations. Most of these studies pertain to Newtonian fluids. Recently, however, a generalization of Newtonian fluid theory has been introduced which takes care of the presence of couple stresses in fluids. The generalized form of the stress principle of Euler and Cauchy states that the action on any portion of a body enclosed by a surface due to the rest of the body is equipollent to a stress vector t_i and a couple stress vector m_i acting over its surface. The non-polar theory of a fluid is characterized by the conditions $m_i = 0$, $l_i = 0$, where l_i is the body moment per unit mass. There is no reason a priori to make $m_i = 0$. Essentially if m_i is not zero, then the action on one part of the body on its neighbourhood cannot be represented by a force alone, but rather by a force and a couple. The couple stress fluid theory presents models for fluids whose microstructure is mechanically significant. The effect of very small microstructure in a fluid can be felt if the characteristic geometric dimension of the problem considered is of the same order of magnitude as the size of the microstructure. A model for such a fluid has been proposed by Stokes [1].

As an example to illustrate the effects of couple stresses on viscous incompressible fluids, Stokes has solved the problem of channel and Couette flow also. In order to study the effect of a transverse magnetic field on the flow of an electrically conducting, viscous, incompressible fluid with couple stresses, Stokes [2] presented the

0021-9991/85 $3.00

problems of MHD channel and MHD Couette flow. Further extension to his work has been done by Soundalgekar and Aranake [3, 4]. In all these cases only steady state flows are considered and the effects of couple stress parameter and Hartmann number on flow are discussed.

The present study deals with the effect of couple stresses on transient hydromagnetic Poiseuille flow based on the model proposed by Stokes [1]. The induced magnetic field is taken into consideration. The defining partial differential equations, one of the fourth order and the other of the second order, are coupled and an exact analytical solution proves difficult. Hence, a numerical solution using the explicit finite difference scheme is obtained.

2. Basic Equations and Solutions

We solve the start-up transient flow in the x direction of a conducting, incompressible, couple stress fluid between parallel insulating plates of infinite extent, in the presence of a uniform magnetic field B_0 in the z direction. The plates are stationary and flow ensues due to a constant pressure gradient suddenly applied to the system at time $t=0$. The rectangular axes (o, x, y, z) are chosen, the boundary walls are given by $z = \pm h$. The components of velocity are $(u(z, t)\, 0, 0)$ and those of magnetic field $(b(z, t), 0, B_0)$ where b is the induced field.

The governing equations in the absence of body force are

$$\rho \frac{\partial u}{\partial t} = \frac{B_0}{\mu_0} \frac{\partial b}{\partial z} + \mu \frac{\partial^2 u}{\partial z^2} - \frac{\partial p}{\partial x} - \eta \frac{\partial^4 u}{\partial z^4}, \tag{2.1}$$

$$\frac{\partial b}{\partial t} = B_0 \frac{\partial u}{\partial z} + \frac{1}{\sigma \mu_0} \frac{\partial^2 b}{\partial z^2}, \tag{2.2}$$

with the initial and boundary conditions

$$u(z, 0) = 0, \qquad b(z, 0) = 0, \tag{2.3}$$

$$u(-h, t) = 0, \qquad u(h, t) = 0, \tag{2.4}$$

$$b(-h, t) = 0, \qquad b(h, t) = 0, \tag{2.5}$$

$$\frac{\partial^2 u}{\partial z^2}(-h, t) = 0, \qquad \frac{\partial^2 u}{\partial z^2}(h, t) = 0, \tag{2.6}$$

where

$-\partial p/\partial x = P =$ constant pressure gradient,

$\eta =$ couple stress parameter,

$\mu_0 =$ permeability of free space,

μ = coefficient of viscosity of the fluid,

ρ = density of the fluid,

σ = electrical conductivity of the fluid.

The effect of couple stresses enter through the last term in (2.1). The boundary conditions (2.4) are due to no slip condition at the plates. Since the plates are insulating, the magnetic field at the plates vanishes, giving the boundary condition (2.5). Since couple stress vanishes at the plates, then when introduced into the defining equations, it gives (2.6).

Defining the following non-dimensional quantities:

$$U=\frac{u}{\frac{h^2}{2\mu}P},\qquad Z=\frac{z}{h},\qquad M=B_0h\left(\frac{\sigma}{\mu}\right)^{1/2},\qquad R_{\mathrm{m}}=\frac{\mu_0\sigma h^3P}{2\mu},$$

$$B=\frac{b}{B_0R_{\mathrm{m}}},\qquad T=\frac{vt}{h^2},\qquad \frac{\eta}{\mu}=l^2,\qquad K=\frac{l}{h}=\frac{\sqrt{\eta/\mu}}{h},\qquad P_m=\mu_0\sigma v,$$

where

M = Hartmann number,

R_{m} = magnetic Reynolds number,

P_{m} = magnetic Prandtl number,

$v=\frac{\mu}{\rho}$ = kinematic viscosity of the fluid.

The above equations reduce to

$$\frac{\partial U}{\partial T}=M^2\frac{\partial B}{\partial Z}+\frac{\partial^2 U}{\partial Z^2}+2-K^2\frac{\partial^4 U}{\partial Z^4}, \tag{2.7}$$

$$P_{\mathrm{m}}\frac{\partial B}{\partial T}=\frac{\partial U}{\partial Z}+\frac{\partial^2 B}{\partial Z^2}, \tag{2.8}$$

with the initial and boundary conditions

$$U(Z,0)=0,\qquad B(Z,0)=0, \tag{2.9}$$

$$U(-1,T)=0,\qquad U(1,T)=0, \tag{2.10}$$

$$B(-1,T)=0,\qquad B(1,T)=0, \tag{2.11}$$

$$\frac{\partial^2 U}{\partial Z^2}(-1,T)=0,\qquad \frac{\partial^2 U}{\partial Z^2}(1,T)=0. \tag{2.12}$$

The non-dimensional couple stress parameter $K=l/h$ is the ratio of the material characteristic length $l=\sqrt{\eta/\mu}$ to half the distance between the plates. Couple stress fluid theory has been used as a model for blood flow in small arteries. Notable contributions have been made by Valanis and Sun [5], Kline, Allen, and DeSilva [6], Kline and Allen [7], Chaturani and Upadhya [8], Chaturani [9, 10], Chaturani and Kaloni [11], and Chaturani and Rathod [12]. In the flow of blood (which is also a conducting fluid) through the arteries, couple stress parameter K is the ratio of the characteristic length l to the radius r of the artery. Since artery radii are different, l is kept constant and r varied to yield different K. Values of K are discussed in references cited above. Equations (2.7) and (2.8) are coupled and an analytical treatment proves difficult. Hence, for different values of K, P_m, and M, the above equations are solved numerically using the explicit finite difference scheme where the derivatives are replaced by their finite difference [13, 14], as follows:

$$\begin{aligned}
\frac{\partial U}{\partial T}&=\frac{U_{i,j+1}-U_{i,j}}{\Delta T},\\
\frac{\partial B}{\partial T}&=\frac{B_{i,j+1}-B_{i,j}}{\Delta T},\\
\frac{\partial U}{\partial Z}&=\frac{U_{i+1,j}-U_{i-1,j}}{2\Delta Z},\\
\frac{\partial B}{\partial Z}&=\frac{B_{i+1,j}-B_{i-1,j}}{2\Delta Z},\\
\frac{\partial^2 U}{\partial Z^2}&=\frac{U_{i+1,j}-2U_{i,j}+U_{i-1,j}}{(\Delta Z)^2},\\
\frac{\partial^2 B}{\partial Z^2}&=\frac{B_{i+1,j}-2B_{i,j}+B_{i-1,j}}{(\Delta Z)^2},\\
\frac{\partial^4 U}{\partial Z^4}&=\frac{U_{i+2,j}-4U_{i+1,j}+6U_{i,j}-4U_{i-1,j}+U_{i-2,j}}{(\Delta Z)^4},
\end{aligned}\tag{2.13}$$

where a network of grid points is first established throughout the region $-1\leqslant Z\leqslant 1$ and $0\leqslant T$ with grid spacings ΔZ and ΔT along Z and T, (i,j) representing a grid point.

Rewriting (2.7) and (2.8) in their finite difference form and rearranging we get

$$\begin{aligned}
U_{i,j+1}=U_{i,j}&+\frac{\Delta T}{2\Delta Z}M^2(B_{i+1,j}-B_{i-1,j})\\
&+\frac{\Delta T}{(\Delta Z)^2}(U_{i+1,j}-2U_{i,j}+U_{i-1,j})\\
&+2\Delta T-\frac{\Delta T}{(\Delta Z)^4}K^2(U_{i+2,j}-4U_{i+1,j}+6U_{i,j}-4U_{i-1,j}+U_{i-2,j})
\end{aligned}\tag{2.14}$$

$$B_{i,j+1} = B_{i,j} + \frac{\Delta T}{2\Delta Z} \frac{1}{P_m} (U_{i+1,j} - U_{i-1,j})$$

$$+ \frac{\Delta T}{(\Delta Z)^2} \frac{1}{P_m} (B_{i+1,j} - 2B_{i,j} + B_{i-1,j}), \tag{2.15}$$

where $i = 0, 1, 2, 3, \ldots, N$ and $j = 0, 1, 2, 3, \ldots$.

If all the $U_{i,j}$ and $B_{i,j}$ are known at time level j, (2.14) and (2.15) enable calculation of $U_{i,j+1}$ and $B_{i,j+1}$ at time level $(j+1)$ directly.

For the boundary points $i = 0$, $i = N$ we have

$$\begin{aligned} U(0,j) &= U(N,j) = 0, \\ B(0,j) &= B(N,j) = 0, \end{aligned} \tag{2.16}$$

$$\frac{\partial^2 U}{\partial Z^2}(0,j) = \frac{\partial^2 U}{\partial Z^2}(N,j) = 0,$$

i.e.,

$$U_{-1,j} = 2U_{0,j} - U_{1,j}$$

$$U_{N+1,j} = 2U_{N,j} - U_{N-1,j}.$$

From the initial condition we have

$$U(i,0) = B(i,0) = 0 \qquad (i = 1, 2, 3, \ldots, N). \tag{2.17}$$

Starting with (2.16) and (2.17) and using (2.14) and (2.15), U and B can evidently be obtained at all grid points, advancing from one time step to another.

3. Stability of the Finite Difference Equations

Since the explicit scheme is used, the largest time step consistent with stability should be known.

For stability of Eqs. (2.14) and (2.15), we proceed thus. Let U_S and B_S be the solutions of Eqs. (2.7) and (2.8) under steady state conditions. Let U_T and B_T be their solution under transient conditions. Putting $U = U_S - U_T$ and $B = B_S - B_T$ in (2.7) and (2.8) we get

$$\frac{\partial U_T}{\partial T} = M^2 \frac{\partial B_T}{\partial Z} + \frac{\partial^2 U_T}{\partial Z^2} - K^2 \frac{\partial^4 U_T}{\partial Z^4}, \tag{3.1}$$

$$P_m \frac{\partial B_T}{\partial T} = \frac{\partial U_T}{\partial Z} + \frac{\partial^2 B_T}{\partial Z^2}. \tag{3.2}$$

Since stability implies the choice of ΔT so that the solution may be bounded, if the stability of (3.1) and (3.2) can be established, then (2.14) and (2.15) become stable also.

Following the procedure of Carnahan *et al.* [15], the general terms of the Fourier expansion for U_T and B_T at arbitrary time $t=0$ are both $e^{i\alpha Z}$ apart from a costant, where α is a positive costant and $i=\sqrt{-1}$. At a time T later, these terms will become

$$\begin{aligned} U_T &= \psi(T)\, e^{i\alpha Z} \\ B_T &= \phi(T)\, e^{i\alpha Z}. \end{aligned} \tag{3.3}$$

Substituting (3.3) in (3.1) and (3.2) expressed in their finite difference form and denoting the values of ψ and ϕ after one time step as ψ' and ϕ', and simplifying, we get

$$\begin{aligned} \psi' &= (1+A-E)\,\psi + M^2 D\phi, \\ \phi' &= (D/P_m)\,\psi + (1+A/P_m)\,\phi, \end{aligned} \tag{3.4}$$

where

$$A = \frac{2\Delta T}{(\Delta Z)^2}(\cos\alpha\Delta Z - 1) = 2a(\cos\alpha\Delta Z - 1),$$

$$D = \frac{\Delta T}{\Delta Z}(i\sin\alpha\Delta Z) = di\sin\alpha\Delta Z,$$

$$\begin{aligned} E &= \frac{2K^2\Delta T}{(\Delta Z)^4}(\cos 2\alpha\Delta Z - 4\cos\alpha\Delta Z + 3), \\ &= 2K^2 e(\cos 2\alpha\Delta Z - 4\cos\alpha\Delta Z + 3), \end{aligned}$$

and

$$a = \frac{\Delta T}{(\Delta Z)^2}, \qquad d = \frac{\Delta T}{\Delta Z}, \qquad e = \frac{\Delta T}{(\Delta Z)^4}.$$

Expressed in matrix notation (3.4) becomes

$$\begin{bmatrix} \psi' \\ \phi' \end{bmatrix} = \begin{bmatrix} (1+A-E) & M^2 D \\ D/P_m & 1+A/P_m \end{bmatrix} \begin{bmatrix} \psi \\ \phi \end{bmatrix},$$

i.e., $G' = FG$.

For stability, the modulus of each eigenvalue of the amplification matrix F should not exceed unity. The eigenvalues λ_1 and λ_2 are

$$\lambda_{1,2}=\frac{1}{2}\left[\left(2+A+\frac{A}{P_m}-E\right) \pm\left\{\left(2+A+\frac{A}{P_m}-E\right)^2-4(1+A-E)(1+A/P_m)+\frac{4M^2D^2}{P_m}\right\}^{1/2}\right] \tag{3.5}$$

We consider the following cases:

(i) $\cos\alpha\Delta Z=-1$, then $A=-4a$, $D=0$, $E=16K^2e$,

(ii) $\cos\alpha\Delta Z=+1$, then $A=0$, $D=0$, $E=0$,

(iii) $\sin\alpha\Delta Z=\pm1$, then $A=-2a$, $D=\pm id$, $E=4K^2e$

Simplifying (3.5) when $D=0$, we get

$$\lambda_1=1+A-E$$

$$\lambda_2=1+A/P_m.$$

Case (i). When $A=-4a$, $E=16K^2e$, the following are the stability conditions to be satisfied:

$$\left|1-\frac{4\Delta T}{(\Delta Z)^2}-16K^2\frac{\Delta T}{(\Delta Z)^4}\right|\leqslant 1 \tag{3.6}$$

$$\left|1-\frac{4}{P_m}\frac{\Delta T}{(\Delta Z)^2}\right|\leqslant 1. \tag{3.7}$$

Case (ii). When $A=0$, $E=0$,

$$|\lambda_1|=1, \qquad |\lambda_2|=1$$

thereby satisfying stability requirements.

Case (iii). When $D=\pm id$, $A=-2a$, $E=4K^2e$,

$$\lambda_{1,2}=\left(1-a-\frac{a}{P_m}-2K^2e\right)\pm\left\{\left(-a+\frac{a}{P_m}-2K^2e\right)^2-\frac{d^2M^2}{P_m}\right\}^{1/2},$$

and the stability conditions to be satisfied are

$$\left|1-\frac{\Delta T}{(\Delta Z)^2}-\frac{\Delta T}{P_m(\Delta Z)^2}-2K^2\frac{\Delta T}{(\Delta Z)^4} \pm\left[\left\{\frac{-\Delta T}{(\Delta Z)^2}+\frac{\Delta T}{P_m(\Delta Z)^2}-\frac{2K^2\Delta T}{(\Delta Z)^4}\right\}^2-\left(\frac{\Delta T}{\Delta Z}\right)^2\frac{M^2}{P_m}\right]^{1/2}\right|\leqslant 1. \tag{3.8}$$

The values of ΔT, ΔZ, P_m, K, and M are chosen so that conditions (3.6), (3.7), and (3.8) are satisfied, thereby assuring the stability of the explicit finite difference scheme used.

4. Discussion and Conclusions

The values of dimensionless velocity U and dimensionless induced magnetic field B are computed as functions of Z and T for different values of the couple stress parameter K, Hartmann number M, and magnetic Prandtl number P_m.

To study the nature of variation, the following values of $P_m = 1.0$, $M = 5.0$, and $K = 0.04$ are considered. The effect of these dimensionless parameters on U and B are studied for different dimensionless times T.

Figures (1–4) are profiles of U for different T, M, P_m, and K. We notice from Fig. 1 that velocity increases as time progresses and attains a steady state at $T = 0.775$. These values compare favourably with known steady state velocity values [2], the difference at maximum value being 0.5%. The effect of the Hartmann number from Fig. 2 is to retard velocity as in conventional MHD flows. Figure 3 indicates the effect of magnetic Prandtl number which increases the velocity. An increase in couple stress parameter K results in a decrease in velocity which is marked for larger Hartmann numbers. The plot shown in Fig. 4 is made for $M = 25$. In all cases the profiles are symmetric as in the case of fluids without couple stress.

Figures (5–7) indicate the variation of dimensionless magnetic field B with Z for $K = 0.04$, for different T, M, and P_m. We find $B = 0$ at the origin. It increases and decreases in magnitude to attain zero at both plates. Since the average value of B is zero, the variation of B is positive from $Z = -1$ to 0 and negative from $Z = 0$ to 1. The induced field decreases with increase in M and P_m. The induced field profiles are skew symmetric as in the case of fluids without couple stress.

The conclusions that couple stress retards flow and that this retardation is more marked for higher Hartmann numbers agree with those of Stokes [2]. The present study also highlights the variation of velocity and induced field with time and the magnetic Prandtl number in addition to the Hartmann number and couple stress parameters.

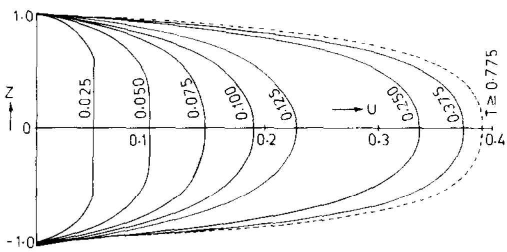

FIG. 1. Velocity profiles for different T when $P_m = 1.0$, $M = 5.0$, $K = 0.04$. The $T \geq 0.775$ curve represents steady state.

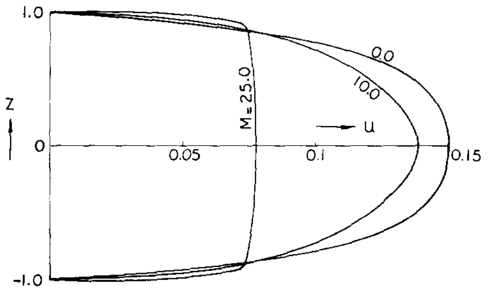

FIG. 2. Velocity profiles for different M when, $P_m = 1.0$, $K = 0.04$, $T = 0.075$.

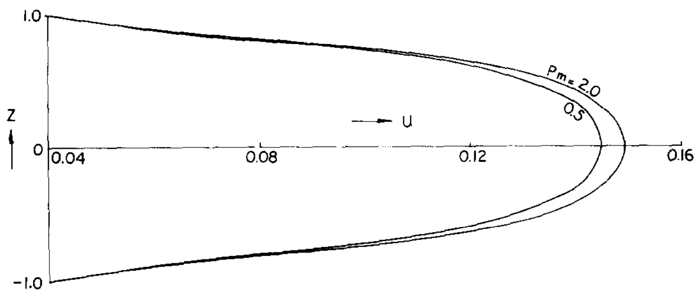

FIG. 3. Velocity profiles for different P_m, when $K = 0.04$, $M = 5.0$, $T = 0.075$.

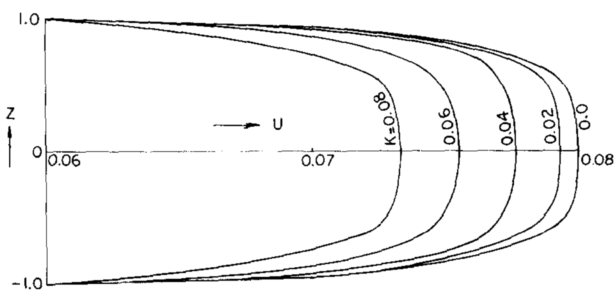

FIG. 4. Velocity profiles for different K, when $P_m = 1.0$, $T = 0.075$, $M = 25$.

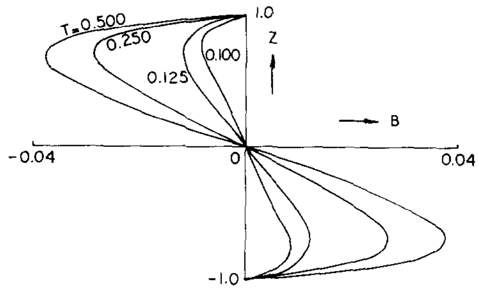

FIG. 5. Induced magnetic field profiles for different T, when $P_m = 1.0$, $M = 5.0$, $K = 0.04$.

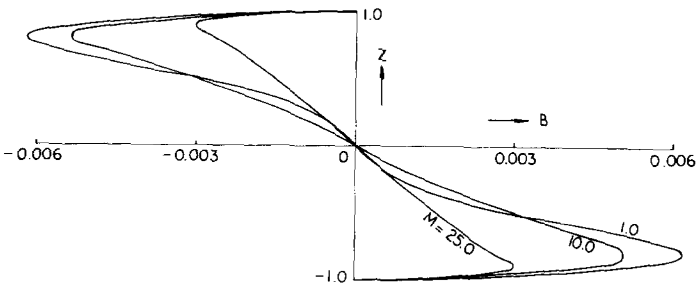

FIG. 6. Induced magnetic field profiles for different M, when $P_m = 1.0$, $K = 0.04$, $T = 0.075$.

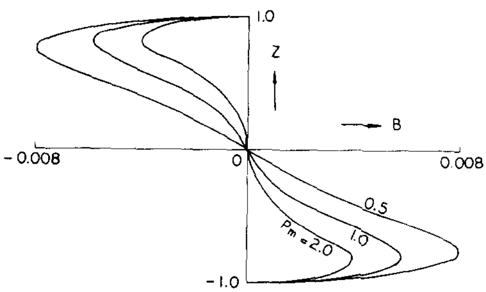

FIG. 7. Induced magnetic field profiles for different P_m, when $K = 0.04$, $M = 5.0$, $T = 0.075$.

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